

SYMMETRIES OF ORDER FOUR ON K3 SURFACES

MICHELA ARTEBANI AND ALESSANDRA SARTI

ABSTRACT. We study automorphisms of order four on K3 surfaces. The symplectic ones have been first studied by Nikulin, they are known to fix six points and their action on the K3 lattice is unique. In this paper we give a classification of the purely non-symplectic automorphisms by relating the structure of their fixed locus to their action on cohomology, in the following cases: the fixed locus contains a curve of genus $g > 0$; the fixed locus contains at least a curve and all the curves fixed by the square of the automorphism are rational. We give partial results in the other cases. Finally, we classify non-symplectic automorphisms of order four with symplectic square.

INTRODUCTION

Let X be a K3 surface over \mathbb{C} with an order four automorphism. Such automorphism acts on the one-dimensional vector space $H^{2,0}(X)$ of holomorphic two-forms of X either as the identity, minus the identity or as the multiplication by $\pm i$. Accordingly, the automorphism is called symplectic, with symplectic square or purely non-symplectic. Symplectic automorphisms of finite order have been investigated by several authors, their fixed locus contains only isolated points (six if the order is four) and their action on the K3 lattice $H^2(X, \mathbb{Z})$ is known to be independent on the surface (cf. [10, 11, 19]). Non-symplectic automorphisms have been classified in [17, 19] (see also [30] for a survey on the topic) and the fixed locus has been identified if the order is prime in [2, 3, 27]. In [26] Taki classified order four non-symplectic automorphisms acting as the identity on the Picard lattice of the surface. Moreover, Schütt [23] studied the special case when the transcendental lattice of the surface has rank four.

This paper deals mainly with purely non-symplectic automorphisms of order four under the assumption that their square is the identity on the Picard lattice. By the Torelli type theorem, this holds for the generic element of the family of K3 surfaces having an order four non-symplectic automorphism with a given action on the K3 lattice. We give a complete classification of the fixed locus of such automorphisms when it either contains a curve of genus $g > 0$ or it contains a curve of genus 0 and all the curves fixed by the square of the automorphism are rational. Moreover, we provide partial results for the remaining cases and we give several examples. We also consider the case when the automorphism has symplectic square: we prove that its fixed locus is empty and its invariant lattice has rank 6.

Date: January 12, 2013.

2010 *Mathematics Subject Classification.* Primary 14J28; Secondary 14J50, 14J10.

Key words and phrases. non-symplectic automorphism, K3 surface.

The first author has been partially supported by Proyecto FONDECYT Regular 2009, N. 1090069 and by Proyecto FONDECYT Regular 2011, N. 1110249.

The study of such automorphisms and their fixed locus is interesting also in relation with the Borcea-Voisin construction of Calabi-Yau varieties and the investigation of Mirror-Symmetry (cf. [5, 29]). In fact Borcea and Voisin consider the product between a K3 surface with a non-symplectic automorphism of order 2, 3, 4 or 6, and an elliptic curve with an automorphism of the same order. A resolution of the quotient variety is then a Calabi-Yau threefold. In [9] Garbagnati used some non-symplectic automorphism of order four to give examples of Calabi-Yau threefolds using this construction.

We now give a short description of the paper's sections. Let σ be a non-symplectic automorphism of order four on a K3 surface X .

In section 1 we give a general description of the fixed locus of σ . In case it is purely non-symplectic, it is the disjoint union of n points, k smooth rational curves and possibly a smooth curve of genus g . By means of Lefschetz's formulas we provide two relations, between the invariants n, k, g and the ranks of the eigenspaces of σ^* on the lattice $H^2(X, \mathbb{Z})$. If σ has symplectic square, we prove that the fixed locus is empty and such ranks are uniquely determined.

In section 2 we study elliptic fibrations $\pi : X \rightarrow \mathbb{P}^1$ such that σ preserves each fiber of π . In Corollary 1 the configuration of the singular fibers, which are of Kodaira type III, I_0^* or III^* , is related to the structure of the fixed locus of σ .

In section 3 we assume that σ fixes pointwisely an elliptic curve E . In Theorem 3.1 we describe the singular fibers of the elliptic fibration with fiber E and the corresponding structure of the fixed locus of σ .

In section 4 and 5 we classify the case when σ contains a curve of genus $g > 1$ in its fixed locus or a rational curve and σ^2 fixes only rational curves.

In section 6 we assume that σ is the identity on the Picard lattice and we give an independent proof of [26, Proposition 4.3].

In section 7 we consider the case when σ only fixes isolated points and we provide families of examples.

In section 8, we study the case when σ only fixes isolated points and rational curves, and σ^2 fixes a curve of genus $g \geq 1$.

Acknowledgements: We warmly thank Bert van Geemen and Alice Garbagnati for useful discussions.

1. THE FIXED LOCUS

Let X be a K3 surface with a *non-symplectic* automorphism σ of order four, i.e. such that the action of σ^* on the vector space $H^{2,0}(X) = \mathbb{C}\omega_X$ of holomorphic two-forms is not trivial. We will call the automorphism *purely non-symplectic* if $\sigma^*\omega_X = \pm i\omega_X$. Otherwise $\sigma^*\omega_X = -\omega_X$ and σ^2 is a symplectic involution.

We will denote by r, l, m the rank of the eigenspace of σ^* in $H^2(X, \mathbb{Z})$ relative to the eigenvalues 1, -1 and i respectively. Moreover, let

$$S(\sigma) = \{x \in H^2(X, \mathbb{Z}) \mid \sigma^*(x) = x\},$$

$$S(\sigma^2) = \{x \in H^2(X, \mathbb{Z}) \mid (\sigma^2)^*(x) = x\}, \quad T(\sigma^2) = S(\sigma^2)^\perp \cap H^2(X, \mathbb{Z}).$$

Observe that $r = \text{rk } S(\sigma)$, $r + l = \text{rk } S(\sigma^2)$ and $2m = \text{rk } T(\sigma^2)$.

Proposition 1. *Let σ be a purely non-symplectic automorphism of order four on a K3 surface X . Then:*

- $\text{Fix}(\sigma^2)$ is either the disjoint union of two elliptic curves or the disjoint union of a smooth curve C of genus $g \geq 0$ and j smooth rational curves;
- $\text{Fix}(\sigma) \subset \text{Fix}(\sigma^2)$ is the disjoint union of smooth curves and n isolated points.

Moreover, the following relations hold:

$$n = 2\alpha + 4, \quad \alpha = \frac{r - l - 2}{4} = \frac{10 - l - m}{2},$$

where $\alpha = \sum_{C_i \subset \text{Fix}(\sigma)} (1 - g(C_i))$.

Proof. Since σ is purely non-symplectic, then σ^2 is a non-symplectic involution. By [20, Theorem 4.2.2] or [3, Theorem 4.1] the fixed locus of σ^2 is either empty, the disjoint union of two elliptic curves or the disjoint union of a curve of genus $g \geq 0$ and smooth rational curves. The action of σ at a point in $\text{Fix}(\sigma)$ can be locally diagonalized as follows (see [19, §5]):

$$A_{4,0} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{4,1} = \begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix}.$$

In the first case the point belongs to a smooth fixed curve, while in the second case it is an isolated fixed point. We will apply holomorphic and topological Lefschetz's formulas to obtain the last two relations in the statement. The Lefschetz number of σ is

$$L(\sigma) = \sum_{j=0}^2 (-1)^j \text{tr}(\sigma^* | H^j(X, \mathcal{O}_X)) = 1 - i,$$

since σ^* acts as multiplication by i on $H^{2,0}(X)$. By [4, p. 567] one obtains:

$$L(\sigma) = \frac{n}{\det(I - \sigma^* | T_P)} + \frac{1+i}{(1-i)^2} \sum_i (1 - g(C_i)) = \frac{n}{\det(I - A_{4,1})} + \alpha \frac{1+i}{(1-i)^2},$$

where n is the number of isolated fixed points, P is an isolated fixed point, T_P denotes the tangent space at P and C_i are the curves in the fixed locus. Comparing the two formulas for $L(\sigma)$ we obtain the relation $n = 2\alpha + 4$. In particular this implies that the fixed locus of σ (and thus that of σ^2) is not empty. We consider now the topological Lefschetz fixed point formula

$$\chi(\text{Fix}(\sigma)) = \sum_{j=0}^4 (-1)^j \text{tr}(\sigma^* | H^j(X, \mathbb{R})) = 2 + \text{tr}(\sigma^* | H^2(X, \mathbb{R})).$$

Since $\text{tr}(\sigma^* | H^2(X, \mathbb{R})) = r - l$, then:

$$\chi(\text{Fix}(\sigma)) = n + 2\alpha = 2 + r - l.$$

Using the relation $n = 2\alpha + 4$, this gives the two expressions for α in the statement. \square

We now provide a similar result in case σ^2 is symplectic.

Proposition 2. *Let σ be a non-symplectic automorphism of order four on a K3 surface X such that σ^2 is symplectic. Then $\text{Fix}(\sigma)$ is empty and $r = 6, l = 8, m = 4$.*

Proof. Since σ has symplectic square, then the local action of σ at a fixed point is of the following type:

$$A_{4,2} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

The holomorphic Lefschetz formula immediately gives that $n = 0$ since $\sigma^* = -id$ on $H^{2,0}(X)$. Moreover, the topological Lefschetz formula gives that $l - r = 2$. Since the invariant lattice of a symplectic involution has rank 14, [19], then $l + r = 14$, so that $l = 8, r = 6$ and $2m = 22 - 14 = 8$. \square

Example 1.1. Consider the following family of quartics surfaces in \mathbb{P}^3 :

$$\begin{aligned} & a_1 x_0^4 + x_0^2(a_2 x_1^2 + a_3 x_2 x_3) + x_0 x_1(a_4 x_2^2 + a_5 x_3^2) + \\ & x_1^2(a_6 x_1^2 + a_7 x_2 x_3) + x_2^2(a_8 x_2^2 + a_9 x_3^2) + a_{10} x_3^4 = 0. \end{aligned}$$

The generic element X_a of the family is a smooth quartic surface, hence a K3 surface, and carries the order four automorphism:

$$\sigma(x_0, x_1, x_2, x_3) = (x_0, -x_1, ix_2, -ix_3),$$

which has no fixed points and whose square fixes the eight intersection points between X_a and the lines $x_0 = x_1 = 0, x_2 = x_3 = 0$. Since the space of matrices in $\mathrm{GL}_3(\mathbb{C})$ commuting with σ has dimension 4, then the family has $10 - 4 = 6$ moduli.

Proposition 3. *Let σ be a non-symplectic automorphism of order four on a K3 surface X . Then $S(\sigma)$ is a hyperbolic sublattice of $\mathrm{Pic}(X)$.*

If σ is purely non-symplectic, then $S(\sigma^2) \subset \mathrm{Pic}(X)$ and it is a 2-elementary lattice with determinant 2^d , such that $\mathrm{rk} S(\sigma^2) = r + l = 10, d = 8$ if σ^2 fixes two elliptic curves and otherwise

$$2g = 22 - r - l - d, \quad 2j = r + l - d.$$

Proof. If $x \in S(\sigma)$, then $(x, \omega_X) = (\sigma^*(x), \sigma^*(\omega_X)) = (x, \alpha \omega_X)$, with $\alpha \neq 1$ since σ is non-symplectic. Thus $x \in \mathrm{Pic}(X) = \omega_X^{-1} \cap H^2(X, \mathbb{Z})$. A similar argument shows that $S(\sigma^2) \subset \mathrm{Pic}(X)$ if σ is purely non-symplectic. Observe that, by [19, Theorem 3.1], the surface X is algebraic. Moreover, it is easy to construct a σ -invariant class with positive self-intersection. This implies that $S(\sigma)$ is a hyperbolic lattice, since $\mathrm{Pic}(X)$ is hyperbolic by Hodge index theorem. The proof that $S(\sigma^2)$ is 2-elementary and the relations in the statement are given in [20, Theorem 4.2.2] or [3, Theorem 4.1]. \square

Remark 1.2. The moduli space of K3 surfaces carrying a purely non-symplectic automorphism of order four with a given action on the K3 lattice is known to be a complex ball quotient of dimension $m - 1$, see [8, §11]. The generic element of such space is a K3 surface such that ω_X is the generic element of an eigenspace of σ^* in $T(\sigma^2) \otimes \mathbb{C}$, so that $\mathrm{Pic}(X) = S(\sigma^2)$. On the other hand, if the automorphism has symplectic square, then the period belongs to the eigenspace where $\sigma^* = -id$, so that $\mathrm{Pic}(X)$ contains $S(\sigma) \oplus T(\sigma^2)$, $\mathrm{rk} \mathrm{Pic}(X) \geq 14$ and, given the action on the K3 lattice, the dimension of the moduli space is equal to 6.

The following result will be useful later.

Lemma 1. *If $x \in S(\sigma^2)$, then $x \cdot \sigma(x)$ is even.*

Proof. If $\sigma(x) = x$, then the statement is obvious since $H^2(X, \mathbb{Z})$ is an even lattice. Otherwise, since x belongs to $S(\sigma^2)$, it is of the form $x = \frac{a+b}{n}$ for some positive integer n where $a \in S(\sigma)$ and b belongs to its orthogonal complement in $S(\sigma^2)$, where $\sigma^* = -\text{id}$. Thus $x \cdot \sigma(x) = \frac{a^2-b^2}{n^2} = \frac{2a^2}{n^2} - x^2$, which is even. \square

2. ELLIPTIC FIBRATIONS

In this section we will study elliptic fibrations on K3 surfaces carrying a purely non-symplectic automorphism of order four. The following result is proved with an argument contained in the proof of [7, Proposition 2.9].

Proposition 4. *Let σ be an automorphism of a K3 surface X . If the rank of the invariant lattice of σ^* in $H^2(X, \mathbb{Z})$ is ≥ 5 , then there is a σ -invariant elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$.*

Proof. We will denote by $S(\sigma)$ the invariant lattice of σ^* in $H^2(X, \mathbb{Z})$. By [24, Corollary 2, pag. 43], since $\text{rk}(S(\sigma)) \geq 5$, there exists a primitive isotropic vector $x \in S(\sigma)$. After applying a finite number of reflections with respect to (-2) -curves, we obtain a nef class x' which is uniquely determined by x (see [22, §6, Theorem 1]). Observe that x' is primitive and $x'^2 = 0$. It is easy to see that σ^* acts on the orbit of x with respect to the reflection group. Since x' is the unique nef member in the orbit and any automorphism preserves nefness, then $\sigma^*(x') = x'$. The morphism associated to x' is a σ -invariant elliptic fibration on X . \square

Let $\pi : X \rightarrow \mathbb{P}^1$ be a σ -invariant elliptic fibration such that any of its fibers is invariant for σ and contains at least a fixed point. The last assumption is not necessary if the fibration is jacobian: indeed if σ is fixed points free on the generic fiber, then it acts as a translation on it and it can be easily proved, by writing explicitly a holomorphic 2-form, that the automorphism would be symplectic. If σ has order four, then the generic fiber of π contains two fixed points for σ and four fixed points of σ^2 . Thus π has two bisections (not necessarily irreducible): a curve $E_\sigma \subset \text{Fix}(\sigma)$ and a curve $E_{\sigma^2} \subset \text{Fix}(\sigma^2)$. We now describe the singular fibers of the elliptic fibration and the action of σ on them.

Proposition 5. *Let X be a K3 surface with a non-symplectic order four automorphism σ and $\pi : X \rightarrow \mathbb{P}^1$ be an elliptic fibration such that σ preserves each fiber of π and has a fixed point on it. Then the singular fibers of π are of the following Kodaira types:*

- *III: $R_1 \cup R_2$, where either*
 - a) the R_i 's are exchanged by σ , E_{σ^2} intersects each R_i at one point and E_σ intersects in $R_1 \cap R_2$ or*
 - b) the R_i 's are σ -invariant, E_σ intersects each R_i at one point and E_{σ^2} intersects in $R_1 \cap R_2$.*
- *I_0^* : $2R_1 + R_2 + R_3 + R_4 + R_5$, where either*
 - a) R_2, R_3 are σ -invariant (intersected by E_σ) and R_4, R_5 are exchanged by σ (intersected by E_{σ^2}) or*
 - b) R_2, \dots, R_5 are permuted by σ , E_σ and E_{σ^2} intersect R_1 .*
- *III*: $R_1 + 2R_2 + 3R_3 + 4R_4 + 2R_5 + 3R_6 + 2R_7 + R_8$, where either*
 - a) σ preserves each irreducible component of the fiber, $R_2, R_4, R_7 \subset \text{Fix}(\sigma)$, E_σ intersects R_1, R_8 and E_{σ^2} intersects R_5 or*

- b) σ preserves each irreducible component of the fiber, $R_4 \subset \text{Fix}(\sigma)$, R_2, R_7 contain two isolated fixed points, E_σ intersects R_1, R_8 , E_{σ^2} intersects R_5 or
c) σ exchanges the two branches of the fiber, E_{σ^2} intersects R_1, R_8 and E_σ intersects R_5 .

Proof. By the previous argument, the restriction of σ to the generic fiber of π has order four and two fixed points. Thus any smooth fiber of π has j -invariant equal to 1. By the Kodaira classification it follows that the singular fibers of π are either of type I_0^* , III , or III^* . We now analyze the possible actions of σ on these fibers.

If F is a reducible fiber of type I_0^* , then the component R_1 is clearly σ -invariant. Observe that R_1 is not fixed by σ , since otherwise each R_i , $i = 2, \dots, 5$, should contain a fixed point for σ in the intersection with either E_{σ^2} or E_σ . This is absurd because σ exchanges the two (distinct) points in $F \cap E_{\sigma^2}$. Thus σ has either order two or four on R_1 . If $\sigma^2 = \text{id}$ on R_1 , then each R_i , $i = 2, \dots, 5$ contains a fixed point of σ^2 in the intersection with either E_{σ^2} or E_σ , thus we are in case I_0^*a). If σ has order 4 on R_1 , then σ permutes the curves R_i , $i = 2, \dots, 5$ and R_1 contains two fixed points for σ in the intersection with E_σ and E_{σ^2} , giving case I_0^*b).

If F is a fiber of type III^* , then R_4 and R_5 are clearly σ -invariant. If σ preserves each irreducible component of F , then $R_4 \subset \text{Fix}(\sigma)$ (since it contains 3 fixed points) and, since E_{σ^2} contains at most a fixed point, E_σ intersects R_1, R_8 and E_{σ^2} intersects R_5 . The curves R_2 and R_7 are either contained in $\text{Fix}(\sigma)$ or contain each two isolated fixed points. These give the cases III^*a) and b) respectively. Otherwise, if σ exchanges the two branches of the fiber, then $\sigma^2 = \text{id}$ on R_4 , E_{σ^2} intersects R_1 and R_8 in two points exchanged by σ and E_σ intersects R_5 . The case of a fiber of type III can be discussed in a similar way. \square

We will denote by g_σ and g_{σ^2} the genus of E_σ and E_{σ^2} respectively, by n the number of isolated points in $\text{Fix}(\sigma)$, by k the number of smooth rational curves in $\text{Fix}(\sigma)$ and by a the number of smooth rational curves in $\text{Fix}(\sigma^2)$ exchanged by σ .

Corollary 1. *Under the hypotheses of Proposition 5, we have the following possibilities for the invariants defined above.*

- If E_σ is irreducible and E_{σ^2} is reducible (hence it is the union of two smooth rational curves exchanged by σ):

g_σ	n	k	a	reducible fibers
3	0	0	1	$8IIIa$)
2	2	0	1	$6IIIa$) + I_0^*a)
		2	2	$5IIIa$) + III^*c)
1	4	0	1	$4IIIa$) + $2I_0^*a$)
		4	2	$3IIIa$) + I_0^*a) + III^*c)
		4	3	$2IIIa$) + $2III^*c$)
0	6	1	1	$2IIIa$) + $3I_0^*a$)
	6	1	2	$IIIa$) + $2I_0^*a$) + III^*c)
	6	1	3	I_0^*a) + $2III^*c$)

- If E_σ is reducible and E_{σ^2} is irreducible: $a = 0$ and

g_{σ^2}	n	k	reducible fibers
3	8	2	$8IIIb)$
2	8	2	$6IIIb) + I_0^*a)$
	10	3	$5IIIb) + III^*b)$
1	8	2	$4IIIb) + 2I_0^*a)$
	10	3	$3IIIb) + I_0^*a) + III^*b)$
	12	4	$2IIIb) + 2III^*b)$
0	8	2	$2IIIb) + 3I_0^*a)$
	10	3	$IIIb) + 2I_0^*a) + III^*b)$
	12	4	$I_0^*a) + 2III^*b)$

- If E_σ and E_{σ^2} are both irreducible:

g_σ	g_{σ^2}	n	k	a	reducible fibers
2	0	2	0	0	$6IIIIa) + 2IIIIb)$
1	1	4	0	0	$4IIIIa) + 4IIIIb)$
		4	0	0	$4I_0^*b)$
1	0	6	1	0	$3IIIIa) + I_0^*b) + III^*b)$
		6	1	0	$4IIIIa) + IIIb) + III^*b)$
		4	0	0	$4IIIIa) + 2IIIIb) + I_0^*a)$
		4	0	1	$3IIIIa) + 2IIIIb) + III^*c)$
0	2	6	1	0	$2IIIIa) + 6IIIIb)$
		6	1	0	$4IIIIb) + 2I_0^*b)$
0	1	8	2	0	$2IIIIa) + 3IIIIb) + III^*b)$
		6	1	0	$2IIIIa) + 4IIIIb) + I_0^*a)$
		6	1	1	$IIIIa) + 4IIIIb) + III^*c)$
0	0	10	3	0	$2IIIIa) + 2IIII^*b)$
		8	2	0	$2IIIIa) + IIIb) + I_0^*a) + III^*b)$
		8	2	1	$IIIIa) + IIIb) + III^*b) + III^*c)$
		6	1	0	$2IIIIa) + 2IIIIb) + 2I_0^*a)$
		6	1	1	$IIIIa) + 2IIIIb) + I_0^*a) + III^*c)$
		6	1	2	$2IIIIb) + 2IIII^*c)$

- If both E_σ and E_{σ^2} are reducible: $n = 8, k = 2, a = 1$ and the reducible fibers are of type $4I_0^*a)$.

Proof. Observe that the restrictions of π to E_σ and to E_{σ^2} are double covers of \mathbb{P}^1 . If E_σ (or E_{σ^2}) is irreducible, then it contains $2g_\sigma + 2$ (or $2g_{\sigma^2} + 2$) ramification points. Since a smooth fiber of π contains exactly 4 fixed points for σ^2 , then such ramification points belong to singular fibers of π , which are classified in Proposition 5. The ramification points of E_σ belong either to a fiber of type $IIIIa)$, $I_0^*b)$ or $III^*c)$. On the other hand, the ramification points of E_{σ^2} belong either to a fiber of type $IIIb)$, $I_0^*b)$, $III^*a)$ or $III^*b)$. This implies that $g_\sigma \leq 3$ (or $g_{\sigma^2} \leq 3$) since otherwise the Euler-Poincaré characteristic of the singular fibers would give at least $e(III)(2g_\sigma + 2) = 3(2g_\sigma + 2) > 24 = e(X)$ (similarly for g_{σ^2}). If E_σ and E_{σ^2} are both irreducible, this implies that $g_\sigma, g_{\sigma^2} \leq 2$. We obtain the tables in the statement by enumerating all cases which are compatible with Proposition 5 and Proposition 1. \square

The following result allows, in some cases, to prove that a given elliptic fibration is σ -invariant.

Proposition 6. *Let X be a K3 surface with an automorphism σ and $\pi : X \rightarrow \mathbb{P}^1$ be an elliptic fibration whose general fiber has class f . If σ^* fixes a class $x \in \text{Pic}(X)$ with $x^2 > 0$, then*

$$(f \cdot \sigma^*(f))x^2 \leq 2(x \cdot f)^2.$$

Moreover, if in addition π is jacobian and there is a section of π not intersecting x , the following holds

$$x^2 \leq \frac{2(x \cdot f)^2}{f \cdot \sigma^*(f) + 1}.$$

Proof. Let M be the sublattice of $\text{Pic}(X)$ generated by x and $f + \sigma^*(f)$. Its intersection matrix has negative determinant $\det(M) = 2(x^2(f \cdot \sigma^*(f)) - 2(x \cdot f)^2) \leq 0$ by Hodge index theorem. This gives the first inequality.

The second inequality follows from a similar argument with the lattice generated by $x, f + \sigma^*(f)$ and the class of the section not intersecting x . \square

Theorem 2.1. *Let σ be a purely non-symplectic automorphism of order 4 on a K3 surface X such that $\text{Pic}(X) = S(\sigma^2) \cong U \oplus R$, where R is a direct sum of root lattices. Then X carries a jacobian elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$ which is σ^2 -invariant, has reducible fibers described by R and a unique section E .*

The involution σ^2 acts as an involution on the simple components of the reducible fibers of π and on the fibers of types I_N^, III^*, II^* as in Figure 1, where σ^2 acts identically on dotted components and as an involution on the other ones.*

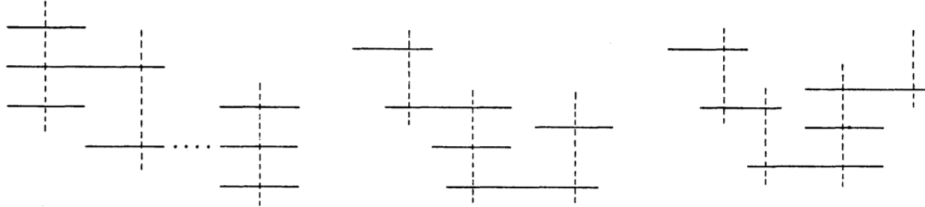


FIGURE 1. Action of σ^2 on reducible fibres of types I_N^*, III^*, II^*

Moreover, if $\text{Fix}(\sigma^2)$ contains a curve C of genus $g > 1$, then:

- a) σ^2 preserves each fiber of π , C intersects the generic fiber at three points and $E \subset \text{Fix}(\sigma^2)$;
- b) π is σ -invariant if $g > 4$;
- c) the genus of a curve in $\text{Fix}(\sigma)$ is ≤ 2 .

Proof. The first half of the statement follows from [14, Lemma 2.1, 2.2]. If σ^2 fixes a curve C of genus $g > 1$, then this curve is transversal to the fibers of π . This implies that σ^2 preserves each fiber of π and has 4 fixed points on it: one on E and three on C (because C intersects each fiber in at least two points and there are no other sections). This proves a).

Let x be the class of C and f be the class of a fiber of π . If $f \neq \sigma^*(f)$, then $f \cdot \sigma^*(f) \geq 2$. It follows from Proposition 6 that $2g - 2 = x^2 \leq \frac{2(x \cdot f)^2}{f \cdot \sigma^*(f) + 1} \leq 6$,

proving *b*). Observe that, if σ fixes a curve C of genus $g > 1$, then $f \neq \sigma^*(f)$ since otherwise the generic fiber would contain at most 2 fixed points by σ . Moreover in this case $f \cdot \sigma^*(f) \geq 4$ since each fiber contains at least 3 fixed points and the intersection $f \cdot \sigma^*(f)$ is even by Lemma 1. This implies $g \leq 2$ by Proposition 6 and proves *c*). \square

3. $\text{Fix}(\sigma)$ CONTAINS AN ELLIPTIC CURVE

We now assume that σ fixes an elliptic curve C . In this case the K3 surface X has an elliptic fibration $\pi_C : X \rightarrow \mathbb{P}^1$ having C as a smooth fiber. Observe that all curves fixed by σ^2 , since they are disjoint from C , are contained in the fibers of π_C . In particular the genus of a fixed curve is ≤ 1 so that $\alpha \geq 0$. We will now classify the reducible fibers of π_C . For the following result, see also [20, §4.2].

Theorem 3.1. *Let σ be a purely non-symplectic order four automorphism on a K3 surface X with $\text{Pic}(X) = S(\sigma^2)$ and $\pi_C : X \rightarrow \mathbb{P}^1$ be an elliptic fibration with a smooth fiber $C \subset \text{Fix}(\sigma)$. Then σ preserves π_C and acts on its basis as an order four automorphism with two fixed points corresponding to the fiber C and a fiber C' which is either smooth, of Kodaira type I_{4M} or IV^* . The corresponding invariants of σ are given in Table 1.*

m	r	l	n	k	a	type of C'
5	7	5	4	0	0	I_0 or I_4
4	10	4	6	1	0	I_8 or IV^*
		8	4	0	1	I_8 or IV^*
3	9	7	4	0	2	I_{12}
2	10	8	4	0	3	I_{16}

TABLE 1. The case $g = 1$

Proof. We first observe that σ^2 is not the identity on the basis of π_C , since otherwise it would act as the identity on the tangent space at a point of C , contradicting the fact that σ^2 is non-symplectic. Hence σ has order four on \mathbb{P}^1 and has two fixed points, corresponding to C and another fiber C' . If C' is irreducible, then $\alpha = 0$ and $n = 4$ by Proposition 1, which implies that C' is smooth elliptic and σ has order two on it.

We now assume that C' is reducible. Since $n \geq 4$ by Proposition 1, then C' contains at least two (disjoint) smooth rational curves fixed by σ^2 . This immediately excludes the Kodaira types I_2, I_3, III, IV for C' . Observe that if a component of C' is preserved by σ^2 and it is “external”, i.e. it only intersects one other component, then it is fixed by σ^2 since otherwise it should contain a fixed point outside of any curve fixed by σ^2 .

If C' is of type I_N^* , then either σ^2 preserves each component of C' or σ has order four on them. In the first case the four external components are fixed by σ^2 by the previous remark and the same should hold for the multiplicity two components intersecting them, since they contain four fixed points, giving a contradiction. In the second case σ^2 exchanges two components of C' of multiplicity one. If they intersect the same component of multiplicity two, then this is invariant for σ^2 but contains

at most a fixed point of σ^2 , a contradiction; if they intersect distinct components of multiplicity two, then it can be easily seen that $n = 0$, a contradiction again.

If C' is either of type II^* or III^* , then σ^2 preserves each component, since the set of components has no order four automorphism. As before, this implies that both the central component (of multiplicity 6 and 4 respectively) and the external component (of multiplicity 3 and 2 respectively) intersecting it are fixed by σ^2 , giving a contradiction.

If C' is of type IV^* , then the central component of multiplicity 3 is clearly invariant for σ . If the central component is fixed by σ , then $k = 1$, $a = 0$ and $n = 6$ by Proposition 1. Otherwise two branches of the fiber are exchanged, and the same Proposition gives $k = 0$, $n = 4$, $a = 1$.

Finally we assume that C' is of type I_N , $N \geq 4$. By the previous remark C' contains at least two components fixed by σ^2 , this implies that all components are preserved by σ^2 and a component which is not fixed intersects two fixed ones. Moreover, it follows from Proposition 1 that the number of components of C' in $\text{Fix}(\sigma^2)$ is even, since it equals $k + n/2 + 2a = \alpha + n/2 + 2a = 2 + 2\alpha + 2a$. Thus N is a multiple of four, i.e. C' is of type I_{4M} for some positive integer M . If $a = 0$, then $n = 4M - 2k$. Since $n = 2k + 4$ by Proposition 1, this gives $k = M - 1$ and $n = 2M + 2$. We now prove that the cases $M = 3, 4$ do not exist if $a = 0$.

If $M = 4$, then $r = 16$ by Proposition 1 and 3. The classes of irreducible components of the fiber I_{16} generate a parabolic sublattice of finite index in $S(\sigma)$. This gives a contradiction since $S(\sigma)$ is a hyperbolic lattice by Proposition 3, thus this case does not appear.

If $M = 3$, then $r = 13$ by Proposition 1 and 3. By the latter proposition and by the classification theorem of 2-elementary lattices [20, Theorem 4.3.1] we have that $S(\sigma^2) \cong U \oplus E_8 \oplus D_4 \oplus A_1^2$. By Theorem 2.1 the surface has a σ^2 -invariant elliptic fibration π with a section E fixed by σ^2 , a reducible fiber of type II^* , one of type I_0^* and two of type \tilde{A}_1 . The section E and a subset of the irreducible components of the first two reducible fibers of π give a chain of 11 smooth rational curves which are contained in the fiber of type I_{12} of π_C . Moreover, the remaining components of the two fibers of π give sections of π_C . Let A and B be two simple components of the fiber of type I_0^* and \tilde{A}_1 respectively not intersecting the section E . Let M be the sublattice of $\text{Pic}(X)$ generated by the classes of the components of the fiber of type I_{12} and by the classes of $A \cup \sigma(A)$ and $B \cup \sigma(B)$. Observe that M is a sublattice of $S(\sigma)$ since $a = 0$. An easy computation shows that the intersection matrix of M has determinant equal to $3(a + b - 4c) + 20$, where a and b are the self-intersections of the classes of $A \cup \sigma(A)$ and $B \cup \sigma(B)$ respectively and c is the intersection between A and $\sigma(B)$. Since such determinant is obviously not zero for any a, b, c ($20 \not\equiv 0 \pmod{3}$), then M is a rank 14 sublattice of $S(\sigma)$, giving a contradiction.

If $a = M - 1$, $M \geq 2$, then σ acts as an order two symmetry on the graph of C' , so that $k = 0$ and $n = 4$. Observe that σ can not act as a rotation of I_{4M} because otherwise $n = 0$, contradicting the fact that $n \geq 4$. \square

Example 3.2. We now assume that the fibration $\pi_C : X \rightarrow \mathbb{P}^1$ in Theorem 3.1 has a σ -invariant section. Then a Weierstrass equation for the fibration is the following:

$$y^2 = x^3 + a(t)x + b(t),$$

where $a(t) = ft^8 + at^4 + b$, $b(t) = gt^{12} + ct^8 + dt^4 + e$ and

$$\sigma(x, y, t) = (x, y, it).$$

The fibers preserved by σ are over $0, \infty$ and the action at infinity is

$$(x/t^4, y/t^6, 1/t) \mapsto (x/t^4, -y/t^6, -i/t).$$

The discriminant polynomial of π_C is:

$$\Delta(t) := 4a(t)^3 + 27b(t)^2 = g_1t^{24} + g_2t^{20} + g_3t^{16} + g_4t^{12} + O(t^8),$$

where

$$\begin{aligned} g_1 &= 4f^3 + 27g^2, & g_2 &= 12f^2a + 54gc, & g_3 &= 12f^2b + 54gd + 12fa^2 + 27c^2, \\ g_4 &= 24fab + 4a^3 + 54ge + 54cd. \end{aligned}$$

For a generic choice of the coefficients of $a(t)$ and $b(t)$ the fibration has 24 fibers of type I_1 over the zeros of $\Delta(t)$, σ fixes pointwisely the fiber over 0 and it acts as an involution on the fiber over ∞ (both fibers are smooth). If $g_1 = 0$ the fibration acquires a fiber of type I_4 by a generic choice of the parameters, if $g_1 = g_2 = g_3 = 0$ we generically get a fiber of type I_{12} and for $g_1 = g_2 = g_3 = g_4 = 0$ we get a fiber of type I_{16} . If $g_1 = g_2 = 0$ one gets two possible solutions: if $f = g = 0$ the fibration acquires a fiber of type IV^* , otherwise it gets a fiber of type I_8 .

More examples for the case $g = 1$ will be given in Examples 4.2 and 4.3.

4. $\text{Fix}(\sigma)$ CONTAINS A CURVE OF GENUS > 1

We now assume that $\text{Fix}(\sigma)$ contains a curve C of genus $g > 1$. By Proposition 1 we have

$$\text{Fix}(\sigma^2) = C \cup (E_1 \cup \dots \cup E_k) \cup (F_1 \cup F'_1 \cup \dots \cup F_a \cup F'_a) \cup (G_1 \cup \dots \cup G_{n/2}),$$

$$\text{Fix}(\sigma) = C \cup E_1 \cup \dots \cup E_k \cup \{p_1, \dots, p_n\},$$

where E_i, F_i, G_i are smooth rational curves such that $\sigma(F_i) = F'_i$, $\sigma(G_i) = G_i$ and each G_i contains exactly two isolated fixed points of σ .

Lemma 2. $k \leq r + m - 8$, $l - m = 2a$.

Proof. The curves C , E_i , $F_i \cup F'_i$, G_i are σ -invariant and are orthogonal to each other, thus their classes in $\text{Pic}(X)$ give independent elements in $S(\sigma)$. Then $r = \text{rk}(S(\sigma)) \geq 1 + k + a + n/2$ and this gives the inequality. An easy computation shows that

$$\mathcal{X}(\text{Fix}(\sigma^2)) - \mathcal{X}(\text{Fix}(\sigma)) = 4a.$$

On the other hand, by Proposition 1 and topological Lefschetz fixed point formula applied to σ^2 :

$$\mathcal{X}(\text{Fix}(\sigma)) = 24 - 2m - 2l, \quad \mathcal{X}(\text{Fix}(\sigma^2)) = 24 - 4m.$$

Comparing these equalities, we obtain the statement. \square

Theorem 4.1. *Let X be a $K3$ surface and σ be a purely non-symplectic automorphism of order four on it such that $\text{Pic}(X) = S(\sigma^2)$. If $\text{Fix}(\sigma)$ contains a curve of genus $g > 1$ then the invariants associated to σ are as in Table 2.*

m	r	l	n	k	a	g
7	1	7	0	0	0	3
6	4	6	2	0	0	2
	2	8	0	0	1	3
5	5	7	2	0	1	2
4	6	8	2	0	2	2

TABLE 2. The case $g > 1$

Proof. If $r = \text{rk}(S(\sigma)) \geq 5$ then, by Proposition 4, X carries a σ -invariant elliptic fibration π . If $C \subset \text{Fix}(\sigma)$ has genus $g > 1$, then C is transversal to the fibers of π , so that any fiber of π is preserved by σ and we are in the first two cases of Corollary 1. Thus, if $r \geq 5$ and $g > 1$, then either $g = n = 2$, $k = 0$ and $a \in \{0, 1, 2\}$ or $g = 3$, $n = k = 0$ and $a = 1$. Thus we now assume that $r < 5$. By Proposition 1 and Lemma 2 we are left for (r, k, g, a) with the cases in Table 2 and the cases $(4, 0, 3, 3)$, $(3, 0, 3, 2)$, $(4, 1, 3, 0)$, $(4, 2, 4, 0)$. In any case we can compute $S(\sigma^2)$ (up to isomorphism) by the classification theorem of 2-elementary lattices [20, Theorem 4.3.1]:

(r, k, g, a)	$S(\sigma^2)$
$(4, 0, 3, 3)$	$U \oplus E_8 \oplus D_4$
$(3, 0, 3, 2)$	$U \oplus D_8 \oplus A_1^{\oplus 2}$
$(4, 1, 3, 0)$	$U \oplus D_4 \oplus A_1^{\oplus 4}; U(2) \oplus D_4^{\oplus 2}$
$(4, 2, 4, 0)$	$U \oplus D_4^{\oplus 2}; U \oplus D_6 \oplus A_1^{\oplus 2}$

In the cases when $S(\sigma^2) \cong U \oplus R$, Theorem 2.1 implies that $g \leq 2$, giving a contradiction. We are left with the case $(4, 1, 3, 0)$ and $S(\sigma^2) \cong U(2) \oplus D_4^{\oplus 2}$. By [14, Lemma 2.1, 2.2] X carries a σ^2 -invariant elliptic fibration π with no sections and two reducible fibers of type I_0^* : $2R + R_1 + R_2 + R_3 + R_4$ and $2R' + R'_1 + R'_2 + R'_3 + R'_4$. Since σ^2 fixes the curve C of genus 3, then any fiber of π is preserved by σ^2 . Moreover, since $\sigma^2 = \text{id}$ on $\text{Pic}(X)$, then each smooth rational curve is σ^2 -invariant. This implies that σ^2 fixes R and R' since each of them contains 4 fixed points. Since $k = 1$, one of these two curves is also fixed by σ , we can assume it to be R . The curve C meets the generic fiber in 4 points and it intersects all the R_i 's and the R'_i 's. This implies that the fibration is not σ -invariant, since otherwise the generic fiber should contain only 2 fixed points for σ . Thus we can assume that $\sigma(R_1) \neq R_1$ and we have that $\beta := R_1 \cdot \sigma(R_1) \geq 2$. The sublattice of $S(\sigma)$ generated by the classes of C, R and $R_1 \cup \sigma(R_1)$ has the following intersection matrix

$$M := \begin{pmatrix} 4 & 0 & 2 \\ 0 & -2 & 2 \\ 2 & 2 & -4 + 2\beta \end{pmatrix}.$$

Since $\det(M) = -16\beta + 24 < 0$, we get a contradiction with the fact that $S(\sigma)$ has hyperbolic signature, thus this case does not appear. \square

Example 4.2 (plane quartics). This construction is due to Kondō [15]. Let C be a smooth plane quartic, defined as the zero set of a homogeneous polynomial

$f_4 \in \mathbb{C}[x_0, x_1, x_2]$ of degree four. The fourfold cover of \mathbb{P}^2 branched along C is a K3 surface with equation

$$t^4 = f_4(x_0, x_1, x_2).$$

The covering automorphism

$$\sigma(x_0, x_1, x_2, t) = (x_0, x_1, x_2, it)$$

is a non-symplectic automorphism of order four whose fixed locus is the plane section $t = 0$, which is isomorphic to the curve C . Thus we have $a = n = k = 0$ and $g = 3$. In case C has ordinary double points (i.e. nodes) or cusps, then the fourfold cover X of \mathbb{P}^2 branched along C has rational double points of type A_3 and E_6 at the inverse images of a node and of a cusp of C respectively. The minimal resolution \tilde{X} of X is a K3 surface and the covering automorphism of X lifts to a non-symplectic automorphism of \tilde{X} . If C has a node, then the central component of the exceptional divisor of type A_3 is fixed by σ^2 and contains two fixed points for σ . If C has a cusp, then the exceptional curve is of type E_6 , σ^2 fixes its two simple components and σ exchanges them. Thus, if C is irreducible with x nodes and y cusps, then the invariants of σ are $g = 3 - x - y$, $a = y$, $n = 2x$ and $k = 0$. Taking $(x, y) = (0, 0), (1, 0)$ and $(0, 1)$ we obtain examples for the first, second and fourth case in Table 2. If $(x, y) = (2, 0), (1, 1)$ or $(0, 2)$ we obtain examples with $g = 1$ corresponding to the cases in Table 1 with a fiber C' of type I_4, I_8 (with $(k, n, a) = (0, 4, 1)$) and I_{12} respectively. If C is the union of a cubic and a line we obtain the case in Table 1 with a fiber C' of type IV^* and $(k, n, a) = (1, 6, 0)$. In [1, Proposition 1.7] the lattice $S(\sigma^2)$ has been computed in case C is irreducible and generic with x nodes and y cusps.

Example 4.3 (hyperelliptic genus three curves). Let \mathbb{F}_4 be a Hirzebruch surface and $e, f \in \text{Pic}(\mathbb{F}_4)$ be the classes of the rational curve E with $E^2 = -4$ and the class of a fiber respectively. A smooth curve C with class $2e + 8f$ is a hyperelliptic genus three curve. Let Y be the double cover of \mathbb{F}_4 branched along C and X be the double cover of Y branched along $C \cup R_1 \cup R_2$, where $R_1 \cup R_2$ is the inverse image of the curve E . The surface X is a K3 surface (see [1, §3]) with a non-symplectic automorphism σ of order 4 whose fixed locus is the inverse image of the curve C and exchanges the curves R_i . Observe that σ^2 fixes $C \cup R_1 \cup R_2$. An alternative construction which associates the K3 surface X to the curve C has been given by Kondō [16]. In this case we have $g = 3$, $n = k = 0$ and $a = 1$.

As in the previous example, if C has at most nodes and cusps, then the minimal resolution \tilde{X} of X is again a K3 surface with a non-symplectic automorphism of order 4. If C is irreducible with x nodes and y cusps, then the invariants of σ are $g = 3 - x - y$, $a = y + 1$, $n = 2x$, $k = 0$. Taking C with a node and a cusp we obtain examples for the third and the last case respectively in Table 2. If $(x, y) = (2, 0), (1, 1)$ or $(0, 2)$ we obtain examples with $g = 1$ corresponding to the cases in Table 1 with a fiber C' of type I_8 (with $(k, n, a) = (0, 4, 1)$), I_{12} and I_{16} respectively. In [16, §4.9] the lattice $S(\sigma^2)$ has been computed in case C is irreducible and generic with x nodes and y cusps.

5. $\text{Fix}(\sigma^2)$ ONLY CONTAINS RATIONAL CURVES

In this section we assume that the curves fixed by σ^2 are rational and that at least one of them is fixed by σ .

Theorem 5.1. *Let X be a K3 surface and σ be a purely non-symplectic automorphism of order four on it. If $\text{Fix}(\sigma)$ contains a smooth rational curve and all curves fixed by σ^2 are rational, then the invariants associated to σ are as in Table 3 (where $k+1$ is the number of curves in $\text{Fix}(\sigma)$).*

m	r	l	n	$k+1$	a
4	10	4	6	1	0
3	13	3	8	2	0
	11	5	6	1	1
2	16	2	10	3	0
	14	4	8	2	1
	12	6	6	1	2
1	19	1	12	4	0
	13	7	6	1	3

TABLE 3. The case $g = 0$

Proof. By Proposition 1 and Lemma 2 we have the cases in Table 3 and the cases $(r, k, a) = (17, 2, 1)$ and $(15, 1, 2)$. In both cases the surface X has a σ -invariant jacobian elliptic fibration by [17, Lemma (1.5), (2)]. By Example 5.2 these two cases do not appear since $a \in \{0, 3\}$. \square

Example 5.2 (Vinberg's K3 surface). If $m = 1$, then $S(\sigma^2) = \text{Pic}(X)$ has maximal rank and X is isomorphic to the unique K3 surface with

$$T(X) = T(\sigma^2) \cong \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

since it can be easily proved that, up to isometry, this is the only rank two even positive definite, 2-elementary lattice and it has moreover an order four isometry without fixed vectors. The automorphism group of this K3 surface is known to be infinite and has been computed by Vinberg in [28, §2.4]. In particular, it is known that X is birationally isomorphic to the following quartics in \mathbb{P}^3 :

$$\begin{aligned} x_0^4 &= -x_1x_2x_3(x_1 + x_2 + x_3), \\ x_0^4 &= x_2^2x_3^2 + x_3^2x_1^2 + x_1^2x_2^2 - 2x_1x_2x_3(x_1 + x_2 + x_3), \end{aligned}$$

which are degree four covers of \mathbb{P}^2 branched along the union of four lines in general position and an irreducible quartic with three cusps respectively. The two covering automorphisms $x_0 \mapsto ix_0$ induce non-conjugate order four non-symplectic automorphisms on X : the first one has $a = 0$, $n = 12$ and fixes 4 smooth rational curves (the proper transforms of the lines), the second one has $a = 3$ (coming from the cusps), $n = 6$ and fixes one smooth rational curve (the proper transform of the quartic). These give the last two examples in Table 3.

All elliptic fibrations $\pi : X \rightarrow \mathbb{P}^1$ are known to be jacobian by [13, Theorem 2.3] and have been classified by Nishiyama in [21, Theorem 3.1]. We recall the classification here, where R is the lattice generated by components of reducible fibers not intersecting the zero section and MW is the Mordell-Weil group of π .

No.	R	MW	a
1)	$E_8^{\oplus 2} \oplus A_1^{\oplus 2}$	0	0, 4
2)	$E_8 \oplus D_{10}$	0	0
3)	$D_{16} \oplus A_1^{\oplus 2}$	\mathbb{Z}_2	0, 3
4)	$E_7^{\oplus 2} \oplus D_4$	\mathbb{Z}_2	0, 3, 4
5)	$E_7 \oplus D_{10} \oplus A_1$	\mathbb{Z}_2	—
6)	$A_{17} \oplus A_1$	\mathbb{Z}_3	0
7)	D_{18}	0	0
8)	$D_{12} \oplus D_6$	\mathbb{Z}_2	0, 3
9)	$D_8^{\oplus 2} \oplus A_1^{\oplus 2}$	$(\mathbb{Z}_2)^2$	0, 3, 4
10)	$A_{15} \oplus A_3$	\mathbb{Z}_4	0, 3, 4
11)	$E_6 \oplus A_{11}$	$\mathbb{Z} \oplus \mathbb{Z}_3$	0, 3
12)	$D_6^{\oplus 3}$	$(\mathbb{Z}_2)^2$	3
13)	$A_9^{\oplus 2}$	\mathbb{Z}_5	0

TABLE 4. Elliptic fibrations of Vinberg's K3 surface

In each case we will compute the possible values taken by a . We will apply the height formula [25, Theorem 8.6] and the notation therein. Moreover, we will apply Theorem 2.1 to determine the number of fixed curves of σ^2 inside the reducible fibers.

- 1) All curves fixed by σ^2 are contained in the reducible fibers. The automorphism σ either preserves the E_8 fibers or it exchanges them, giving $a = 0$ or $a = 4$ respectively.
- 2) The reducible fibers contain exactly 8 curves fixed by σ^2 and the two remaining fixed curves of σ^2 are transversal to the fibers, one of them is the unique section, the other is a 3-section. This implies that both are σ -invariant and σ preserves all components of D_{10} , giving $a = 0$.
- 3) The reducible fibers contain exactly 7 curves fixed by σ^2 . The remaining fixed curves of σ^2 are transversal to the fibers and give two sections s_0, s_1 (assume s_0 to be the zero section) and a 2-section. The translation by the order two element in the Mordell-Weil group gives rise to a symplectic automorphism of order two. Since such automorphism has 8 fixed points by [19], then it acts on the fiber D_{16} as a reflection with respect to the central component, i.e. s_0 and s_1 intersect the fiber in simple components not meeting the same component. This implies that σ acts on the components of D_{16} either as the identity or as a reflection with respect to the central component, giving $a = 0$ and $a = 3$ respectively.
- 4) As before, the reducible fibers only contain 7 curves fixed by σ^2 and the fibration has two sections s_0, s_1 and a 2-section fixed by σ^2 . If σ preserves each fiber, then either s_0, s_1 are fixed by σ or they are exchanged giving $a = 0$ and $a = 3$ respectively by Corollary 1. Otherwise, if σ has order two on the basis of π , then it exchanges the two fibers of type E_7 , so that either $a = 3$ or $a = 4$.
- 5) Since the fibration has three reducible fibers of distinct types, then σ acts as the identity on \mathbb{P}^1 . This is not possible by Proposition 5, thus this fibration can not be σ -invariant.
- 6) The fiber of type A_{17} contains 9 curves fixed by σ^2 and is clearly σ -invariant.

Since $18 \not\equiv 0 \pmod{4}$, then $a = 0$.

7) The fiber of type D_{18} contains 8 curves fixed by σ^2 , the remaining two curves fixed by σ^2 give a section and a 3-section. Thus $a = 0$.

8) The reducible fibers contain 7 curves fixed by σ^2 , the remaining fixed curves of σ^2 give two sections s_0, s_1 . By the height formula, s_0 and s_1 meet simple components intersecting distinct components in the D_{12} fiber and intersecting the same component in the D_6 fiber. Thus either $a = 0$ or $a = 3$.

9) The reducible fibers contain 6 curves fixed by σ^2 and the fibration has four sections s_0, s_1, s_2, s_3 fixed by σ^2 . Observe that the elliptic fibration has two fibers of type D_8 , two of type I_2 and no other singular fibers (since the Euler characteristic of X is 24). By the height formula, we can assume that s_1 intersects $\Theta_1^1, \Theta_2^2, \Theta_3^3, \Theta_4^4$, s_2 intersects $\Theta_2^1, \Theta_1^2, \Theta_3^3, \Theta_4^4$ and s_3 intersects $\Theta_3^1, \Theta_3^2, \Theta_0^3, \Theta_0^4$. Observe that σ either preserves the fibers D_8 and exchanges the fibers I_2 , or it exchanges both pairs of reducible fibers, or it exchanges the fibers D_8 and it preserves the fibers I_2 . In the first case $a = 0, 3$ or 4 , according to the action of σ on the sections s_i 's. In the second case $a = 4$ (observe that the 4 fixed points by σ are contained in two smooth fibers). The last case does not appear since otherwise σ should preserve all components of the fibers I_2 , which gives a contradiction since the components intersecting s_1, s_2 contain no fixed points for σ .

10) In this case all fixed curves by σ^2 are contained in the two reducible fibers, thus σ has order 4 on \mathbb{P}^1 . Observe that the fibration has four sections $s_i, i = 0, 1, 2, 3$, preserved by σ^2 , so that each of them intersects the curves fixed by σ^2 . This remark and the height formula imply that we can assume that s_1 intersects Θ_8^1, Θ_0^2 , s_2 intersects Θ_4^1, Θ_2^2 and s_3 intersects $\Theta_{12}^1, \Theta_2^2$. This implies that σ either preserves all components of the reducible fibers ($a = 0$), or it acts on the sections as a permutation $(s_2 s_3)$ or $(s_0 s_1)$ ($a = 3$), or as a permutation of type $(s_0 s_1)(s_2 s_3)$ ($a = 4$).

11) In this case all fixed curves by σ^2 are contained in the two reducible fibers, thus σ has order 4 on \mathbb{P}^1 . By the height formula we can assume that s_1 intersects Θ_1^1, Θ_4^2 and s_2 intersects Θ_2^1, Θ_8^2 . This implies that σ either preserves all components of reducible fibers ($a = 0$), or it acts on the sections as a transposition ($a = 3$).

12) The reducible fibers contain 6 curves fixed by σ^2 , the remaining fixed curves give four sections $s_i, i = 0, 1, 2, 3$. Observe that σ has order two on \mathbb{P}^1 , it exchanges two fibers of type D_6 and it preserves the third one. By the height formula we can assume that s_1 intersects $\Theta_1^1, \Theta_3^2, \Theta_3^3$, s_2 intersects $\Theta_3^1, \Theta_3^2, \Theta_1^3$ and s_3 intersects $\Theta_3^1, \Theta_1^2, \Theta_2^3$. This implies that σ acts on the sections as a transposition, so that $a = 3$.

13) All fixed curves by σ^2 are contained in the two reducible fibers and σ preserves the two fibers of type I_{10} (since otherwise $a = 5$, which is not possible), so that $a = 0$.

Example 5.3. Let \tilde{X} and C as in Example 4.2. Taking C with 3 nodes or with 2 nodes and a cusp we obtain examples for the first and third case in Table 3 respectively. If C is the union of two conics (or the union of a line and a nodal cubic), the union of a conic and two lines or the union of a line and a cuspidal cubic we obtain examples for the second, fourth and fifth case in the table respectively. Finally, as mentioned in the previous example, if C is the union of four lines or has 3 cusps, we obtain the last two cases in the table.

Similarly, if \tilde{X} and C are as in Example 4.3 and C has 3 nodes, 2 nodes and a cusp or 1 node and two cusps, we obtain examples for the third, sixth and the last case in Table 3 respectively.

6. THE CASE $l = 0$

In this section we will assume that $l = 0$, i.e. σ acts as the identity on $S(\sigma^2)$.

Proposition 7. *Let σ be a non-symplectic automorphism on a K3 surface X such that $l = 0$. Then $\text{Fix}(\sigma)$ is the disjoint union of smooth rational curves and points, $r = 2 \pmod{4}$ and $a = 0$.*

Proof. Assume that σ fixes a curve of genus $g \geq 1$, then we get a contradiction by Theorem 3.1 and by Lemma 2 (since it says that $-m = 2a$). Observe that $r = 2 \pmod{4}$ by Proposition 1. If a is not zero, then there are two rational curves F_1, F'_1 fixed by σ^2 such that $\sigma(F_1) = F'_1$. If f_1, f'_1 are their classes in $\text{Pic}(X)$, then $\sigma^*(f_1 - f'_1) = f'_1 - f_1$ and $f_1 - f'_1$ is not zero, contradicting $l = 0$. \square

Lemma 3. *Let L be a lattice which is the direct sum of lattices isomorphic to $U \oplus U$, $U \oplus U(2)$, E_8 or D_{4k} , $k \geq 1$. Then L has an isometry τ with $\tau^2 = -\text{id}$ acting trivially on $A_L = L^\vee/L$.*

Proof. It is known that the Weyl group of a lattice isometric to either E_8 or D_{4k} , $k \geq 1$, contains an isometry τ with $\tau^2 = -\text{id}$ acting trivially on A_L , [6]. An isometry τ of $U \oplus U$ as in the statement can be defined as follows:

$$\tau : e_1 \mapsto e_2, e_2 \mapsto -e_1, e_3 \mapsto e_4, e_4 \mapsto -e_3,$$

where e_1, e_2 and e_3, e_4 are the natural generators of the first and the second copy of U . Such an action can be defined similarly on $U \oplus U(2)$. \square

By Proposition 7 the fixed locus of σ and of its square are as follows:

$$\text{Fix}(\sigma^2) = C \cup (E_1 \cup \dots \cup E_k) \cup (G_1 \cup \dots \cup G_{n_1/2}),$$

$$\text{Fix}(\sigma) = E_1 \cup \dots \cup E_k \cup \{p_1, \dots, p_n\},$$

where C is a curve of genus g , E_i, G_i are smooth rational curves such that $\sigma(G_i) = G_i$ and each G_i contains exactly two isolated fixed points of σ . We will denote by $n_2 = n - n_1$ the number of isolated fixed points of σ contained in C .

Theorem 6.1. *Let σ be a non-symplectic automorphism on a K3 surface X such that $S(\sigma^2) = S(\sigma) = \text{Pic}(X)$. Then the invariants of the fixed locus of σ , the lattices $S(\sigma^2)$ and $T(\sigma^2)$ (up to isomorphism) appear in the following table.*

Proof. By Proposition 1 we have that

$$\chi(\text{Fix}(\sigma^2)) - \chi(\text{Fix}(\sigma)) = 2 - 2g - n_2 = -2m.$$

Moreover, since σ acts on C as an order two automorphism with n_2 fixed points, then $2g - 2 - n_2 \geq -4$, i.e. $g \geq \frac{n_2}{2} - 1$ by Riemann-Hurwitz formula. These remarks, together with Proposition 1 and the classification theorem of 2-elementary even lattices [20, Theorem 4.3.1] give a list of possible cases, the ones appearing in the table and some more with $S(\sigma^2)$ isomorphic to one of the following lattices:

$$U \oplus A_1^{\oplus 4}, U \oplus D_6 \oplus A_1^{\oplus 2}, U \oplus D_4 \oplus A_1^{\oplus 4}, U \oplus E_8 \oplus A_1^{\oplus 4}, U \oplus E_8 \oplus E_7 \oplus A_1, \\ U \oplus E_7 \oplus A_1, (2) \oplus A_1.$$

m	r	n_1	n_2	k	g	$S(\sigma^2)$	$T(\sigma^2)$
10	2	2	2	0	10	U	$U \oplus U \oplus E_8^{\oplus 2}$
	2	0	4	0	9	$U(2)$	$U \oplus U(2) \oplus E_8^{\oplus 2}$
8	6	2	4	1	7	$U \oplus D_4$	$U \oplus U \oplus E_8 \oplus D_4$
	6	0	6	1	6	$U(2) \oplus D_4$	$U \oplus U(2) \oplus E_8 \oplus D_4$
6	10	6	2	2	6	$U \oplus E_8$	$U \oplus U \oplus E_8$
	10	4	4	2	5	$U(2) \oplus E_8$	$U \oplus U(2) \oplus E_8$
	10	2	6	2	4	$U \oplus D_4^{\oplus 2}$	$U \oplus U \oplus D_4^{\oplus 2}$
	10	0	8	2	3	$U(2) \oplus D_4^{\oplus 2}$	$U \oplus U(2) \oplus D_4^{\oplus 2}$
4	14	6	4	3	3	$U \oplus E_8 \oplus D_4$	$U \oplus U \oplus D_4$
	14	4	6	3	2	$U(2) \oplus E_8 \oplus D_4$	$U \oplus U(2) \oplus D_4$
2	18	10	2	4	2	$U \oplus E_8^{\oplus 2}$	$U \oplus U$
	18	8	4	4	1	$U(2) \oplus E_8^{\oplus 2}$	$U \oplus U(2)$

TABLE 5. The case $l = 0$

In the first five cases X has a σ -invariant jacobian elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$ with more than two reducible fibers by Theorem 2.1. Since σ fixes > 2 points in the basis of π , then it preserves each fiber of π . This gives a contradiction since σ should have two fixed points in each fiber while the fibration has a unique section fixed by σ .

We now show that the case $S(\sigma^2) \cong (2) \oplus A_1$ does not appear. Let e, f be the generators of $S(\sigma^2)$ with $e^2 = 2, f^2 = -2, e \cdot f = 0$. The class e is nef and the associated morphism is a degree two map $\pi : X \rightarrow \mathbb{P}^2$ which is the minimal resolution of the double cover branched along an irreducible plane sextic S with a node at the image of the curve with class $\pm f$. Since $\sigma^*(e) = e$, then σ induces a projectivity $\bar{\sigma}$ of \mathbb{P}^2 with $\bar{\sigma}^2 = \text{id}$ (since it fixes $\pi(C) = S$). This implies that, up to a choice of coordinates, a birational model of X and σ are given as follows:

$$X : w^2 = f_6(x_0, x_1, x_2), \quad \sigma(x_0, x_1, x_2, w) = (-x_0, x_1, x_2, iw),$$

where f_6 is a homogeneous degree six polynomial with $\sigma(f_6) = -f_6$ and singular at one point. Observe that such polynomial f_6 contains $x_0 = 0$ as a component, giving a contradiction. If $S(\sigma) \cong U \oplus E_7 \oplus A_1 \cong (2) \oplus A_1 \oplus E_8$, then X is the minimal resolution of the double cover of \mathbb{P}^2 branched along an irreducible sextic with a node and a triple point of type E_8 . We can exclude this case by an argument similar to the previous one.

If $T(\sigma^2)$ is any lattice appearing in Table 5, then it carries an isometry τ with $\tau^2 = -\text{id}$ and acting trivially on the discriminant group by Lemma 3. It follows that the isometries $\text{id}_{S(\sigma^2)}$ and τ glue to give an order four isometry ρ of L_{K3} . By the Torelli-type Theorem [18, Theorem 3.10] there exists a K3 surface X with a non-symplectic automorphism σ of order four such that $\sigma^* = \rho$ up to conjugacy. Thus all cases in the table do exist. \square

Remark 6.2. In all cases of Table 5 the lattices $S(\sigma^2)$ and $T(\sigma^2)$ are 2-elementary even lattices with $x^2 \in \mathbb{Z}$ for all $x \in L^\vee$. A lattice theoretical proof of this fact and an alternative proof of Theorem 5 are given by Taki in [26, Proposition 2.4].

Example 6.3. Consider the elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$ in Weierstrass form given by

$$y^2 = x^3 + a(t)x + b(t),$$

where a is an even polynomial and b is an odd polynomial in t . Observe that it carries the order four automorphism

$$(x, y, t) \mapsto (-x, iy, -t).$$

For generic coefficients X is a K3 surface, the fixed locus of σ^2 is the union of the curve of genus 10 defined by $y = 0$ and the section $x = z = 0$ and σ fixes four points on them (in the fibers over $t = 0, \infty$). It follows by propositions 1, 3 and [20, Theorem 4.3.1] that $S(\sigma^2) \cong U$, $r = 2$, $l = 0$ and $m = 10$.

A geometric construction of this family of K3 surfaces can be given as follows. Let Y be the Hirzebruch surface \mathbb{F}_4 and $e, f \in \text{Pic}(Y)$ be the classes of the (-4) -curve and of a fiber respectively. Observe that a section of $-2K_Y = 4e + 12f$ is the disjoint union of the (-4) -curve E and a curve C with class $3e + 12f$ (e is in the base locus of $4e + 12f$). The generic such C is smooth and the double cover $p : X \rightarrow Y$ branched along $C \cup E$ is a K3 surface. We denote by \tilde{C} and \tilde{E} the pull-backs of C and E by p , observe that $g(\tilde{C}) = 10$ and $g(\tilde{E}) = 0$. We will consider the affine coordinates $t = u_1/u_2$ and $x = v_1/v_2$, where u_1, u_2 give a basis of $H^0(Y, f)$ and $v_1 \in H^0(Y, e + 4f)$, $v_2 \in H^0(Y, e)$ are non-zero sections. Let $\iota \in \text{Aut}(Y)$ be the involution on Y given by $(t, x) \mapsto (-t, -x)$ and let $f = 0$ be the equation of \tilde{C} in local coordinates. If $f(-t, -x) = -f(t, x)$, then $f(t, x) = x^3 + a(t)x + b(t)$ where a is an even, degree 8 and b is an odd, degree 11 polynomial in t . In this case ι lifts to an order 4 automorphism σ on X , in local coordinates:

$$X : y^2 = f(t, x), \quad \sigma(t, x, y) = (-t, -x, iy).$$

The ruling of Y induces a jacobian elliptic fibration on X having \tilde{C} as a trisection and \tilde{E} as a section, such that its Weierstrass equation and the action of σ on it are clearly the same as the ones given for the fibration π at the beginning of this example.

Example 6.4. Consider the involution $\iota : ((x_0, y_0), (x_1, y_1)) \mapsto ((x_0, -x_1), (y_0, -y_1))$ of $\mathbb{P}^1 \times \mathbb{P}^1$ and let f be a bihomogeneous polynomial of degree $(4, 4)$ such that $\iota(f) = -f$. If $C = \{f = 0\}$ is a smooth curve, then the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along C

$$X : w^2 = f(x_0, x_1, y_0, y_1)$$

is a K3 surface and carries the order four automorphism:

$$\sigma : (((x_0, x_1), (y_0, y_1)), w) \mapsto (((x_0, -x_1), (y_0, -y_1)), iw).$$

The fixed locus of σ^2 is the genus 9 curve defined by $w = 0$ and σ fixes 4 points on it. It follows by propositions 1, 3 and [20, Theorem 4.3.1] that $S(\sigma^2) \cong U(2)$ (equals the pull-back of $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$), $r = 2$, $l = 0$ and $m = 10$.

7. $\text{Fix}(\sigma)$ ONLY CONTAINS ISOLATED POINTS

Let σ be a purely non-symplectic automorphism of order four on a K3 surface X having only isolated fixed points. It follows from Proposition 1 that $\text{Fix}(\sigma)$ contains exactly four points p_1, \dots, p_4 . Moreover, the fixed locus of σ^2 is as follows:

$$\text{Fix}(\sigma^2) = C \cup (F_1 \cup F'_1) \cup \dots \cup (F_a \cup F'_a) \cup G_1 \cup \dots \cup G_{n_1/2},$$

where each G_i is a smooth rational curve which contains 2 fixed points of σ and C is a smooth genus g curve which contains the remaining $4 - n_1$ fixed points.

Theorem 7.1. *Let σ be a purely non-symplectic automorphism of order 4 having only isolated fixed points on a K3 surface X . Then σ fixes exactly 4 points. Moreover, if $\text{Pic}(X) = S(\sigma^2)$ and $l > 0$, then the invariants of $\text{Fix}(\sigma^2)$ and the lattice $S(\sigma^2)$ appear in Table 6.*

Proof. Since $\alpha = k = 0$, it follows from Proposition 1 that $n = 4$, $r = l + 2$ and $m = 10 - l$. The fixed locus of σ^2 contains $2a + n_1/2$ smooth rational curves and a curve of genus g . Thus by Proposition 3 we obtain $r + l = 11 - g + 2a + n_1/2$.

Observe that the case $l = 1, n_1 = 4$ does not exist since in this case, by Proposition 3 and [20, Theorem 4.3.1] (see Figure 1 in [3]), a curve fixed by σ^2 has genus ≤ 8 .

If $S(\sigma^2) \cong U \oplus R$, where R is a direct sum of root lattices, and $g > 4$, then by Theorem 2.1 the surface X carries a σ -invariant jacobian elliptic fibration π with reducible fibers of type R . The automorphism σ acts as an involution on \mathbb{P}^1 and preserves two fibers F, F' of π . The curve $C \subset \text{Fix}(\sigma^2)$ of genus g intersects each fiber at 3 points and $C \cap F, C \cap F'$ are σ -invariant. This implies that C contains at least two fixed points, i.e. $n_2 \geq 2$. Observe that π has a section fixed by σ^2 and σ -invariant, thus $n_1 \geq 2$. Moreover, $a = 0$ if the rational curves fixed by σ^2 are at most 2, i.e. if $n_1/2 + 2a \leq 2$.

Since C intersects the generic fiber of π in 3 points, then it is trigonal. If $g \geq 3$, this implies that C is not hyperelliptic, hence the canonical morphism of C is an embedding in \mathbb{P}^{g-1} . The involution σ on C is thus induced by an automorphism of the projective space. If $g = 3$, this implies that C is isomorphic to a plane quartic and, since an involution of \mathbb{P}^2 fixes a line, that $n_2 > 0$.

If $(m, r, g, a) = (7, 5, 6, 1)$, then $S(\sigma^2) \cong U \oplus D_6$. Observe that σ^2 fixes the section of π and two irreducible components of the fiber of type D_6 . Since $a = 1$ and the fibration is σ -invariant, then σ should act as a reflection on the fiber D_6 , but this is not possible since the fibration has a unique section.

The cases in the table are then obtained by taking all possible values for l , using the previous equations and remarks and the fact that $n_1 \in \{0, 2, 4\}$. The lattices can be computed by means of propositions 1, 3 and the classification theorem of 2-elementary lattices [20, Theorem 4.3.1]. \square

Example 7.2. Let $C, E \subset \mathbb{F}_4$ as in Example 6.3. If C has rational double points, then the minimal resolution X of the double cover of \mathbb{F}_4 branched along $C \cup E$ is still a K3 surface. If C has two nodes exchanged by ι , then ι lifts to an order four automorphism σ of X such that σ^2 fixes a curve of genus 8 (the pull-back of the proper transform of C) and a smooth rational curve \tilde{E} (the pull-back of the (-4) -curve E). Observe that here $l > 0$ since the exceptional divisors over the two nodes are exchanged by σ . The lattice $S(\sigma^2)$ in this case is isomorphic to $U \oplus A_1^{\oplus 2}$.

If C has two triple points exchanged by ι , then σ^2 fixes the pull-back of the proper transform of the curve C , \tilde{E} and the central components of the resolution trees over the two singular points, which are of type \tilde{D}_4 . In this case $a = 1$ since such components are exchanged by σ . In this case the lattice $S(\sigma^2)$ is isomorphic to $U \oplus D_4^{\oplus 2}$. Similarly, we get examples if the triple points of C are simple singularities of type $D_n (n \geq 6), E_7, E_8$.

m	r	l	n_1	n_2	g	a	$S(\sigma^2)$
9	3	1	2	2	8	0	$U \oplus A_1^{\oplus 2}$ $U(2) \oplus A_1^{\oplus 2}$
8			2	2	6	0	$U \oplus A_1^{\oplus 4} \circ U(2) \oplus D_4$ $U(2) \oplus A_1^{\oplus 4}$
7			2	2	4		$U \oplus A_1^{\oplus 6}$ $U(2) \oplus A_1^{\oplus 6}$
6	6	4	4	0	3	0	$U(2) \oplus D_4^{\oplus 2}$ $U \oplus A_1^{\oplus 8}$ $U(2) \oplus A_1^{\oplus 8}$ $U \oplus D_4 \oplus A_1^{\oplus 4} \circ U(2) \oplus D_4^{\oplus 2}$ $U \oplus D_4^{\oplus 2} \circ U \oplus D_6 \oplus A_1^{\oplus 2}$
5	7	5	4	0	1	0	$U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$ $U \oplus A_1^{\oplus 10}$ $U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$ $U \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$ $U \oplus E_7 \oplus A_1^{\oplus 3}$ $U \oplus E_8 \oplus A_1^{\oplus 2}$
4	8	6	2	2	0	1	$U \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 4}$ $U(2) \oplus D_6^{\oplus 2} \circ U \oplus D_4^{\oplus 3}$ $U(2) \oplus D_6^{\oplus 2} \circ U \oplus D_4^{\oplus 3}$ $U \oplus E_8 \oplus A_1^{\oplus 4} \circ U(2) \oplus D_4 \oplus E_8$ $U \oplus E_8 \oplus D_4$
3	9	7	2	2	0	2	$U \oplus D_6^{\oplus 2} \oplus A_1^{\oplus 2}$ $U(2) \oplus E_7^{\oplus 2}$ $U(2) \oplus E_7^{\oplus 2}$ $U \oplus E_7^{\oplus 2}$
2	10	8	2	2	0	3	$U \oplus E_7^{\oplus 2} \oplus A_1^{\oplus 2} \circ U \oplus D_8^{\oplus 2}$ $U \oplus E_8 \oplus E_7 \oplus A_1 \circ U(2) \oplus E_8^{\oplus 2}$ $U \oplus E_8 \oplus E_7 \oplus A_1 \circ U(2) \oplus E_8^{\oplus 2}$
1	11	9	2	2	0	4	$U \oplus E_8^{\oplus 2} \oplus A_1^{\oplus 2}$

TABLE 6. The case $\alpha = k = 0$, $l > 0$

Considering C with ordinary nodes (up to 10) and triple points exchanged by ι we obtain several examples for the cases in the table with $S(\sigma^2) \cong U \oplus R$.

Similarly, we can construct examples for the cases of type $U(2) \oplus R$ by generalizing Example 6.4 to the case when the curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ has simple singularities. In this way, we obtain examples for the cases in Table 6, excepted the boxed ones.

Example 7.3. Consider the jacobian elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$ defined as follows:

$$y^2 = x(x^2 + a(t^4)x + b(t^4)),$$

where a, b are polynomials of degree 1 and 2 respectively. Observe that π has a 2-torsion section $t \mapsto (0, 0, t)$. The translation by this section gives a symplectic involution τ on X . Moreover, X has the order four non-symplectic automorphisms $\sigma : (x, y, z, t) \mapsto (x, y, z, it)$ and $\sigma' := \sigma \circ \tau$. For generic a, b , π has 8 fibers of type I_2 and 8 fibers of type I_1 . The automorphisms σ and σ' act with order four on \mathbb{P}^1 , preserve the two smooth fibers over $t = 0$ and $t = \infty$ and act as an involution over $t = \infty$. Moreover, σ fixes pointwisely the fiber over $t = 0$, while σ' acts as an order two translation on it.

For special choices of a and b we can obtain examples with reducible fibers of type I_{4M} over $t = 0$ or $t = \infty$. For example, if $a(t^4) = t^4$ and $b(t^4) = 1$, then π has a smooth fiber over $t = 0$ and a fiber of type I_{16} over $t = \infty$. The automorphism σ fixes pointwisely the fiber over $t = 0$ and acts on the fiber over $t = \infty$ as a reflection which leaves invariant the components Θ_0, Θ_8 (see the notation in Example 5.2), giving $k = 0, a = 3, n = 4$. The symplectic involution τ acts over $t = \infty$ as a rotation sending Θ_0 to Θ_8 . Finally, the automorphism σ' acts over $t = 0$ as a translation and over $t = \infty$ as a reflection which leaves invariant the components Θ_4, Θ_{12} , giving $k = 0, a = 3, n_2 = 0, n_1 = 4$ (see Table 6). For more details on this example see also [12, Proposition 4.7].

8. THE OTHER CASES

At this point of the classification the cases left out are those with

$$\text{Fix}(\sigma) = E_1 \cup \dots \cup E_k \cup \{p_1, \dots, p_n\}$$

$$\text{Fix}(\sigma^2) = C \cup (E_1 \cup \dots \cup E_k) \cup (F_1 \cup F'_1 \cup \dots \cup F_a \cup F'_a) \cup (G_1 \cup \dots \cup G_{\frac{n_1}{2}})$$

where C is a curve of genus $g > 0$, $n_2 = n - n_1$ is the number of fixed points on C and we can assume that $k > 0$ and $l > 0$ (recall that we have also $m > 0$). Observe that in this case $\alpha = k$ so $n = 2k + 4$ and $2k = 10 - l - m$ by Proposition 1. In particular, observe that:

$$m + l \equiv 0 \pmod{2} \quad \text{and} \quad m + l \leq 8.$$

On the other hand, by computing the difference $\chi(\text{Fix}(\sigma^2)) - \chi(\text{Fix}(\sigma))$ topologically and using the Lefschetz's formula, one gets the relation

$$2 - 2g - n_2 + 4a = 2l - 2m$$

so that

$$(1) \quad g - 2a = m - l + 1 - \frac{n_2}{2}.$$

Using Hurwitz formula on C we obtain

$$(2) \quad n_2 \leq 2g + 2.$$

By Proposition 3 we also have:

$$(3) \quad g + j = g + 2a + k + \frac{n_1}{2} \leq 11$$

Combining (1) and (3) we get

$$(4) \quad g \leq 5 - k + \frac{m - l}{2} = m$$

$$(5) \quad a \leq 2 - \frac{k}{2} + \frac{n_2}{4} + \frac{(l-m)}{4} \leq 3 + \frac{l-m}{4}$$

where we obtain the last inequality using $n_2 \leq 2k + 4$. Finally observing that $|m-l| \leq 6$ and $k \geq 1$ we get $g \leq 7$ and $a \leq 4$.

Lemma 4. *Assume that $g = g(C) > 1$. Then $g \leq m$ and we are in one of the following cases:*

$m+l$	k	$g \leq$	$a \leq$
4	3	3	2
6	2	5	3
8	1	7	4

Proof. By (4) we have that $g \leq m$, in particular $m \geq 2$. Moreover, by the previous conditions we have that $m+l = 4, 6, 8$. This gives the inequalities for g in the table. If $m+l = 4, 6$ then $|m-l| \leq 2$ and $|m-l| \leq 4$ respectively, so by (5) we get $a \leq 3$ and $a \leq 4$ respectively. Similarly, if $m+l = 8$ then $a \leq 4$.

We show that the case $m+l = 4$, $k = 3$ and $a = 3$ is not possible. By (5) we get $l \geq m$ and by (1) with $n_2 \leq 2k + 4 = 10$ we get $g \leq 2 + (m-l)$. If $l > m$ then $g < 2$, which is not possible. If $l = m$ then we have $g = 2$, hence again $n_2 = 10$ by (1), contradicting (2). The case $m+l = 6$, $a = 4$ can be excluded similarly. \square

Example 8.1. Consider the following elliptic K3 surface $\pi : X \rightarrow \mathbb{P}^1$ in Weierstrass form:

$$y^2 = x^3 - a(t)x, \quad \deg a(t) = 8.$$

with the order four automorphism

$$\sigma(x, y, t) = (-x, iy, t).$$

The automorphism σ fixes the two sections $x = y = 0$, $x = z = 0$ (hence $k \geq 2$), while σ^2 also fixes the curve $C : y = x^2 - a(t) = 0$. The discriminant of the fibration is $\Delta(t) = 4a^3(t)$, hence for a generic polynomial $a(t)$ the fibration has 8 fibers of type *III* (more precisely these are of type *IIIb*). Moreover, the curve C has genus 3 and σ has 8 fixed points on it, so that: $k = 2$, $n_2 = 8$, $a = 0$. One can compute also that $l = 0$, so that this case appears in Table 5. We now study how $a(t)$ can split:

- i) If $a(t) = a_1(t)^2 a_6(t)$, then π has a fiber I_0^* and 6 fibers *IIIb*). In this case the ramification points on C are 6, so that $g(C) = 2$. Here $k = 2$, $n_1 = 2$, $n_2 = 6$, $a = 0$. Thus we have an example in Lemma 4.
- ii) If $a(t) = a_1(t)^2 b_1(t)^2 a_4(t)$ then π two fibers I_0^* and four fibers *IIIb*). Here $k = 2$, $n_1 = 4$, $n_2 = 4$, $a = 0$ and $g = 1$. This case appears in Proposition 8.
- iii) If $a(t) = a_1(t)^2 b_1(t)^2 c_1(t)^2 a_2(t)$ then we have 3 fibers I_0^* and two *IIIb*), so that $k = 2$, $n_1 = 6$, $n_2 = 2$, $a = 0$ and $g = 0$. This case appears in Table 3.
- iv) If $a(t) = a_1(t)^2 b_1(t)^2 c_1(t)^2 d_1(t)^2$ then we have four fibers I_0^* . In this case C splits into the union of two rational curves. In this case we have $k = 2$, $n = 8$, $a = 1$. We are again in one case of Table 3.

For generic $a(t)$ the Mordell-Weil group of X is isomorphic to \mathbb{Z}_2 . The translation by a generator of the group is a symplectic involution ι on X and the composition $\sigma' = \iota \circ \sigma$ is again a non-symplectic automorphism of order 4 on X which fixes C and exchanges the two sections of π .

Example 8.2. In Lemma 4 an easy computation with MAPLE finds 63 possible cases. One can produce some examples with the fixed locus described there starting from Example 6.3 (resp. Example 6.4) and imposing simple singularities on the curve C of genus 10 (resp. genus 9) such that at least one singular point is invariant for ι . For example, one can assume that the curve C in Example 6.3 has an ordinary triple point at an invariant point of ι and two nodes exchanged by ι . The Dynkin diagram of the resolution over the triple point, which is of type \tilde{D}_4 , is σ -invariant: its simple components are preserved and the double component is fixed by σ . Thus $k = 1$, $n_2 = 4$, $n_1 = 2$, $a = 0$ and $g = 5$. Observe that, if C has just an ordinary triple point at an invariant point, then we are in the case $g = 7$, $k = 1$, $n_1 = 2$, $n_2 = 4$ of Table 5. Similar examples can be constructed from Example 6.4.

We now consider the case when σ^2 fixes an elliptic curve.

Proposition 8. *With the previous notation, if $g(C) = 1$ then we are in one of the cases appearing in Table 7.*

m	r	l	n_1	n_2	k	a	type of C'
5	9	3	2	4	1	0	I_4
4	12	2	4	4	2	0	I_8
	10	4	6	0	1	0	
3	15	1	6	4	3	0	I_{12}
	13	3	8	0	2	0	
4	12	2	4	4	2	0	IV^*
	10	4	2	4	1	1	
	10	4	6	0	1	0	

TABLE 7. The case $g = 1, k > 0, l > 0$.

Proof. Using the relations at the beginning of the paragraph one can find the values in the table, we now show that these are the only possibilities.

Since σ preserves C , then there is a σ -invariant elliptic fibration $\pi_C : X \rightarrow \mathbb{P}^1$ with fiber C . Observe that σ has order four on the basis of π_C , since otherwise σ^2 would act as the identity on the tangent space at a point of C . Thus σ has two fixed points on \mathbb{P}^1 , corresponding to the fiber C and a fiber C' of π_C . This implies that all rational curves fixed by σ are contained in C' , so that C' is reducible (since $k > 0$). Observe that σ acts on C either as an involution with four fixed points or as a translation. By Proposition 1 we have $n \geq 6$, so that C' contains at least two fixed points of σ . This excludes the Kodaira types I_2, I_3, III, IV for C' . By similar arguments as in the proof of Theorem 3.1 also the types I_N^*, II^* and III^* can be excluded. If C' has Kodaira type IV^* then σ either preserves each component or it exchanges two branches. In the first case $a = 0$ and either $n = n_1 = 6$ or $n_1 = n_2 = 4$. In the second case $a = 1$, $k = 1$, $n_1 = 2$ and $n_2 = 4$.

We now consider the case when C' is of type I_N , $N \geq 4$. Since C' contains at least a fixed curve for σ , then all components of I_N are preserved by σ , hence $a = 0$. By the previous remarks, since $n_2 = 4$ or 0 , then either $n_1 = 2k$ or $n_1 = 2k + 4$ respectively.

If $n_2 = 4$, then $N = 2k + n_1 = 4k$. For $N > 12$ we get $k > 3$, which gives $m < 3$ by Proposition 1. By the equality (1) we get $m - l = 2$ so $m = 2$ and $l = 0$, contradicting the assumption $l > 0$. Hence we only have the cases in the table.

On the other hand, if $n_2 = 0$ we get $N = 2k + (2k + 4) = 4k + 4$. By equation (1) we get $l = m$ and $k + m = 5$ by Proposition 1. If $N \geq 16$ then $k \geq 3$, which gives $m < 3$ as before. If $m = l = 1$, then $k = 4$, $n_1 = 12$ and C' is of type I_{20} . Since $a = 0$ the lattice $S(\sigma)$ contains the classes of the 20 components of C' , contradicting $r = 19$. If $m = l = 2$, then $k = 3$, $n_1 = 10$ and C' is of type I_{16} . Since $a = 0$ the lattice $S(\sigma)$ contains the classes of the 16 components of C' . This gives a contradiction since $r = 16$ and $S(\sigma)$ is hyperbolic by Proposition 3. \square

Example 8.3. Observe that the elliptic K3 surface $\pi_C : X \rightarrow \mathbb{P}^1$ in Example 3.2 also carries the non-symplectic order four automorphism

$$\sigma'(x, y, t) = (x, -y, it),$$

obtained by composing the automorphism σ defined there with the non-symplectic involution $y \mapsto -y$. Generically, the automorphism σ' acts on the elliptic curve over $t = 0$ as an involution with 4 isolated fixed points and as the identity on the fiber over $t = \infty$. If the fiber over $t = \infty$ is reducible, then σ fixes at most rational curves, in particular $k = \alpha$. Observe that σ' preserves the section at infinity $t \mapsto (0 : 1 : 0; t)$ and has two fixed points on it, one on the fiber over $t = 0$ the other over $t = \infty$. Using this condition and the equality $n = 2\alpha + 4$ given by Proposition 1, one sees that $n = 6$, $k = 1$ if π_C has a fiber of type I_4 , $(k, n, a) = (2, 8, 0)$ for a fiber I_8 , $(3, 10, 0)$ for a fiber I_{12} and $(k, n, a) = (4, 12, 0)$ for a fiber I_{16} . For a fiber of type IV^* there are two possibilities: either σ' fixes two curves, and so we have $(k, n, a) = (2, 8, 0)$, or it acts as a reflection fixing one curve, giving $(k, n, a) = (1, 6, 0)$. The automorphism $\sigma^2 = (\sigma')^2$ is an involution fixing the smooth elliptic curve over $t = 0$ and some rational curves in the singular fiber over $t = \infty$. The cases with the fibers I_4 , I_8 , I_{12} and IV^* give examples with $l \neq 0$ that appear in Table 7. In case there is a fiber of type I_{16} we have instead $l = 0$ (see the last line of Table 5).

REFERENCES

1. M. Artebani, *A compactification of M_3 via K3 surfaces*, Nagoya Math. J. **196** (2009), 1–26. MR 2591089 (2011a:14070)
2. M. Artebani and A. Sarti, *Non-symplectic automorphisms of order 3 on K3 surfaces*, Math. Ann. **342** (2008), no. 4, 903–921. MR 2443767 (2009h:14065)
3. M. Artebani, A. Sarti, and S. Taki, *K3 surfaces with non-symplectic automorphisms of prime order.*, To appear in Math. Z., 2010.
4. M. F. Atiyah and I. M. Singer, *The index of elliptic operators. III*, Ann. of Math. (2) **87** (1968), 546–604.
5. Ciprian Borcea, *K3 surfaces with involution and mirror pairs of Calabi-Yau manifolds*, Mirror symmetry, II, AMS/IP Stud. Adv. Math., vol. 1, Amer. Math. Soc., Providence, RI, 1997, pp. 717–743. MR 1416355 (97i:14023)
6. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray. MR 827219 (88g:20025)
7. I. V. Dolgachev and J. Keum, *Finite groups of symplectic automorphisms of K3 surfaces in positive characteristic*, Ann. of Math. (2) **169** (2009), no. 1, 269–313. MR 2480606 (2009k:14072)

8. I.V. Dolgachev and S. Kondō, *Moduli of K3 surfaces and complex ball quotients*, Arithmetic and geometry around hypergeometric functions, Progr. Math., vol. 260, Birkhäuser, Basel, 2007, pp. 43–100. MR 2306149 (2008e:14053)
9. A. Garbagnati, *New families of Calabi-Yau 3-folds without maximal unipotent monodromy*, Preprint, arXiv:1005.00941.
10. A. Garbagnati and A. Sarti, *Symplectic automorphisms of prime order on K3 surfaces*, J. Algebra **318** (2007), no. 1, 323–350. MR 2363136 (2008j:14070)
11. ———, *Elliptic fibrations and symplectic automorphisms on K3 surfaces*, Comm. Algebra **37** (2009), no. 10, 3601–3631. MR 2561866 (2011a:14076)
12. B. van Geemen and A. Sarti, *Nikulin involutions on K3 surfaces*, Math. Z. **255** (2007), no. 4, 731–753. MR 2274533 (2007j:14057)
13. J. Keum, *A note on elliptic K3 surfaces*, Trans. Amer. Math. Soc. **352** (2000), no. 5, 2077–2086. MR 1707196 (2000j:14059)
14. S. Kondō, *Automorphisms of algebraic K3 surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44** (1992), no. 1, 75–98. MR 1139659 (93e:14046)
15. ———, *A complex hyperbolic structure for the moduli space of curves of genus three*, J. Reine Angew. Math. **525** (2000), 219–232. MR MR1780433 (2001j:14039)
16. ———, *The moduli space of 8 points of \mathbb{P}^1 and automorphic forms*, Algebraic geometry, Contemp. Math., vol. 422, Amer. Math. Soc., Providence, RI, 2007, pp. 89–106. MR MR2296434 (2008h:14027)
17. N. Machida and K. Oguiso, *On K3 surfaces admitting finite non-symplectic group actions*, J. Math. Sci. Univ. Tokyo **5** (1998), no. 2, 273–297. MR 1633933 (99f:14048)
18. Y. Namikawa, *Periods of Enriques surfaces*, Math. Ann. **270** (1985), no. 2, 201–222. MR 771979 (86j:14035)
19. V.V. Nikulin, *Finite groups of automorphisms of Kählerian surfaces of type K3*, Uspehi Mat. Nauk **31** (1976), no. 2(188), 223–224. MR 0409904 (53 #13656)
20. ———, *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. algebrogeometric applications*, Soviet. Math. Dokl. **20** (1979), 1156–1158.
21. K. Nishiyama, *The Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups*, Japan. J. Math. (N.S.) **22** (1996), no. 2, 293–347. MR 1432379 (97m:14037)
22. I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, *Torelli's theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 530–572. MR 0284440 (44 #1666)
23. M. Schütt, *K3 surfaces with non-symplectic automorphisms of 2-power order*, J. Algebra **323** (2010), no. 1, 206–223. MR 2564835 (2011a:14078)
24. J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York, 1973, Translated from the French, Graduate Texts in Mathematics, No. 7. MR 0344216 (49 #8956)
25. T. Shioda, *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Paul. **39** (1990), no. 2, 211–240. MR 1081832 (91m:14056)
26. S. Taki, *Classification of non-symplectic automorphisms on K3 surfaces which act trivially on the Néron Severi lattice*, Preprint, 2010.
27. ———, *Classification of non-symplectic automorphisms of order 3 on K3 surfaces*, Math. Nachr. **284** (2011), 124–135.
28. È. B. Vinberg, *The two most algebraic K3 surfaces*, Math. Ann. **265** (1983), no. 1, 1–21. MR 719348 (85k:14020)
29. C. Voisin, *Mirrors and involutions on K3 surfaces. (Miroirs et involutions sur les surfaces K3.)*, Journées de géométrie algébrique d'Orsay, France, juillet 20-26, 1992. Paris: Société Mathématique de France, Astérisque. 218, 273–323 (1993)., 1993.
30. D-Q. Zhang, *Automorphisms of K3 surfaces*, Proceedings of the International Conference on Complex Geometry and Related Fields (Providence, RI), AMS/IP Stud. Adv. Math., vol. 39, Amer. Math. Soc., 2007, pp. 379–392. MR 2341193 (2008g:14066)

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDAD DE CONCEPCIÓN, CASILLA 160-C, CONCEPCIÓN,
CHILE

E-mail address: `martebani@udec.cl`

LABORATOIRE DE MATHÉMATIQUES ET APPLICATIONS, UMR CNRS 6086, UNIVERSITÉ DE POITIERS,
TÉLÉPORT 2, BOULEVARD MARIE ET PIERRE CURIE, F-86962 FUTUROSCOPE CHASSENEUIL

E-mail address: `sarti@math.univ-poitiers.fr`

URL: `http://www-math.sp2mi.univ-poitiers.fr/~sarti/`